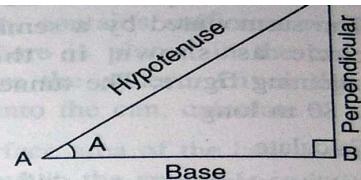


Trigonometrical Ratios :

There are six trigonometrical ratios relating to the three sides of a right-angled triangle (this has already been done by students in Class IX).

For an acute angle of a right-angled triangle :

- (1) **sine** (sin) = $\frac{\text{Perpendicular}}{\text{Hypotenuse}}$ $\Rightarrow \sin A = \frac{BC}{AC}$
- (2) **cosine** (cos) = $\frac{\text{Base}}{\text{Hypotenuse}}$ $\Rightarrow \cos A = \frac{AB}{AC}$
- (3) **tangent** (tan) = $\frac{\text{Perpendicular}}{\text{Base}}$ $\Rightarrow \tan A = \frac{BC}{AB}$
- (4) **cotangent** (cot) = $\frac{\text{Base}}{\text{Perpendicular}}$ $\Rightarrow \cot A = \frac{AB}{BC}$
- (5) **secant** (sec) = $\frac{\text{Hypotenuse}}{\text{Base}}$ $\Rightarrow \sec A = \frac{AC}{AB}$
- (6) **cosecant** (cosec) = $\frac{\text{Hypotenuse}}{\text{Perpendicular}}$ $\Rightarrow \operatorname{cosec} A = \frac{AC}{BC}$



Relations Between Different Trigonometrical Ratios :

1. Reciprocal relations :

Since $\sin A = \frac{\text{perpendicular}}{\text{hypotenuse}}$ and $\operatorname{cosec} A = \frac{\text{hypotenuse}}{\text{perpendicular}}$
 $\Rightarrow \sin A$ and $\operatorname{cosec} A$ are reciprocals of each other

$$\text{i.e. } \sin A = \frac{1}{\operatorname{cosec} A} \quad \text{and} \quad \operatorname{cosec} A = \frac{1}{\sin A}$$

Similarly, (i) $\cos A$ and $\sec A$ are reciprocals of each other

$$\text{i.e. } \cos A = \frac{1}{\sec A} \quad \text{and} \quad \sec A = \frac{1}{\cos A}$$

(ii) $\tan A$ and $\cot A$ are reciprocals of each other

$$\text{i.e. } \tan A = \frac{1}{\cot A} \quad \text{and} \quad \cot A = \frac{1}{\tan A}.$$

2. Quotient relations :

$$\begin{aligned} \text{Since } \sin A &= \frac{\text{perpendicular}}{\text{hypotenuse}} \quad \text{and} \quad \cos A = \frac{\text{base}}{\text{hypotenuse}} \\ \therefore \frac{\sin A}{\cos A} &= \frac{\text{perpendicular}}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{base}} \\ &= \frac{\text{perpendicular}}{\text{base}} = \tan A \end{aligned}$$

$$\text{Similarly, } \frac{\cos A}{\sin A} = \cot A$$

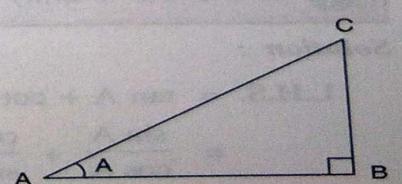
$$\text{Hence, } \tan A = \frac{\sin A}{\cos A} \quad \text{and} \quad \cot A = \frac{\cos A}{\sin A}$$

3. Square relations :

In right-angled triangle ABC, with angle B = 90°;

$$\sin A = \frac{BC}{AC} \quad \text{and} \quad \cos A = \frac{AB}{AC}$$

$$\begin{aligned} \Rightarrow \sin^2 A + \cos^2 A &= \left(\frac{BC}{AC}\right)^2 + \left(\frac{AB}{AC}\right)^2 \\ &= \frac{BC^2 + AB^2}{AC^2} \\ &= \frac{AC^2}{AC^2} = 1 \end{aligned}$$



[As, $AB^2 + BC^2 = AC^2$]

$$\therefore \sin^2 A + \cos^2 A = 1$$

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FORMULA

Similarly,

$$\begin{aligned}
 \text{(i)} \quad 1 + \tan^2 A &= 1 + \left(\frac{BC}{AB}\right)^2 \\
 &= \frac{AB^2 + BC^2}{AB^2} = \frac{AC^2}{AB^2} \quad [\because AB^2 + BC^2 = AC^2] \\
 &= \left(\frac{AC}{AB}\right)^2 = \sec^2 A \quad [\because \sec A = \frac{AC}{AB}]
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad 1 + \cot^2 A &= 1 + \left(\frac{AB}{BC}\right)^2 \\
 &= \frac{BC^2 + AB^2}{BC^2} = \frac{AC^2}{BC^2} \\
 &= \left(\frac{AC}{BC}\right)^2 = \operatorname{cosec}^2 A \quad [\because \operatorname{cosec} A = \frac{AC}{BC}]
 \end{aligned}$$

Hence,

$$\sin^2 A + \cos^2 A = 1; \quad 1 + \tan^2 A = \sec^2 A \quad \text{and} \quad 1 + \cot^2 A = \operatorname{cosec}^2 A.$$

Remember :

(i) $\sin^2 A + \cos^2 A = 1$	$\Rightarrow \sin^2 A = 1 - \cos^2 A$	and	$\cos^2 A = 1 - \sin^2 A$
(ii) $1 + \tan^2 A = \sec^2 A$	$\Rightarrow \sec^2 A - \tan^2 A = 1$	and	$\sec^2 A - 1 = \tan^2 A$
(iii) $1 + \cot^2 A = \operatorname{cosec}^2 A$	$\Rightarrow \operatorname{cosec}^2 A - \cot^2 A = 1$	and	$\operatorname{cosec}^2 A - 1 = \cot^2 A$

PROBLEMS

Prove the following identities :

$$\begin{aligned}
 1. \quad \frac{\sec A - 1}{\sec A + 1} &= \frac{1 - \cos A}{1 + \cos A} \\
 2. \quad \frac{1 + \sin A}{1 - \sin A} &= \frac{\operatorname{cosec} A + 1}{\operatorname{cosec} A - 1} \\
 3. \quad \frac{1}{\tan A + \cot A} &= \cos A \sin A \\
 4. \quad \tan A - \cot A &= \frac{1 - 2 \cos^2 A}{\sin A \cos A} \\
 5. \quad \sin^4 A - \cos^4 A &= 2 \sin^2 A - 1 \\
 6. \quad (1 - \tan A)^2 + (1 + \tan A)^2 &= 2 \sec^2 A \\
 7. \quad \operatorname{cosec}^4 A - \operatorname{cosec}^2 A &= \cot^4 A + \cot^2 A \\
 8. \quad \sec A (1 - \sin A) (\sec A + \tan A) &= 1 \\
 9. \quad \operatorname{cosec} A (1 + \cos A) (\operatorname{cosec} A - \cot A) &= 1 \\
 10. \quad \sec^2 A + \operatorname{cosec}^2 A &= \sec^2 A \cdot \operatorname{cosec}^2 A
 \end{aligned}$$

[2007]

$$\begin{aligned}
 18. \quad \frac{1}{\sec A + \tan A} &= \sec A - \tan A \\
 19. \quad \operatorname{cosec} A + \cot A &= \frac{1}{\operatorname{cosec} A - \cot A} \\
 20. \quad \frac{\sec A - \tan A}{\sec A + \tan A} &= 1 - 2 \sec A \tan A + 2 \tan^2 A \\
 21. \quad (\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 &= 7 + \tan^2 A + \cot^2 A \\
 22. \quad \sec^2 A \cdot \operatorname{cosec}^2 A &= \tan^2 A + \cot^2 A + 2 \\
 23. \quad \frac{1}{1 + \cos A} + \frac{1}{1 - \cos A} &= 2 \operatorname{cosec}^2 A \\
 24. \quad \frac{1}{1 - \sin A} + \frac{1}{1 + \sin A} &= 2 \sec^2 A \\
 25. \quad \frac{\operatorname{cosec} A}{\operatorname{cosec} A - 1} + \frac{\operatorname{cosec} A}{\operatorname{cosec} A + 1} &= 2 \sec^2 A \\
 26. \quad \frac{\sec A}{\sec A + 1} + \frac{\sec A}{\sec A - 1} &= 2 \operatorname{cosec}^2 A \\
 27. \quad \frac{1 + \cos A}{1 - \cos A} &= \frac{\tan^2 A}{(\sec A - 1)^2} \\
 28. \quad \frac{\cot^2 A}{(\operatorname{cosec} A + 1)^2} &= \frac{1 - \sin A}{1 + \sin A} \\
 29. \quad \frac{1 + \sin A}{\cos A} + \frac{\cos A}{1 + \sin A} &= 2 \sec A
 \end{aligned}$$

[2012]

$$30. \frac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$$

$$31. (\cot A - \operatorname{cosec} A)^2 = \frac{1-\cos A}{1+\cos A}$$

$$32. \frac{\operatorname{cosec} A - 1}{\operatorname{cosec} A + 1} = \left(\frac{\cos A}{1+\sin A} \right)^2$$

$$33. \tan^2 A - \tan^2 B = \frac{\sin^2 A - \sin^2 B}{\cos^2 A \cdot \cos^2 B}$$

$$34. \frac{\sin \theta - 2 \sin^3 \theta}{2 \cos^3 \theta - \cos \theta} = \tan \theta \quad [2017]$$

$$35. \frac{\sin A}{1+\cos A} = \operatorname{cosec} A - \cot A$$

$$36. \frac{\cos A}{1-\sin A} = \sec A + \tan A$$

$$37. \frac{\sin A \tan A}{1-\cos A} = 1 + \sec A$$

$$38. (1 + \cot A - \operatorname{cosec} A)(1 + \tan A + \sec A) = 2$$

$$39. \sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$$

$$40. \sqrt{\frac{1-\cos A}{1+\cos A}} = \operatorname{cosec} A - \cot A$$

$$41. \sqrt{\frac{1-\cos A}{1+\cos A}} = \frac{\sin A}{1+\cos A} \quad [2000, 2013]$$

$$42. \sqrt{\frac{1-\sin A}{1+\sin A}} = \frac{\cos A}{1+\sin A}$$

$$43. 1 - \frac{\cos^2 A}{1+\sin A} = \sin A$$

$$44. \frac{1}{\sin A + \cos A} + \frac{1}{\sin A - \cos A} = \frac{2 \sin A}{1 - 2 \cos^2 A} \quad [2002]$$

$$45. \frac{\sin A + \cos A}{\sin A - \cos A} + \frac{\sin A - \cos A}{\sin A + \cos A} = \frac{2}{2 \sin^2 A - 1}$$

$$46. \frac{\cot A + \operatorname{cosec} A - 1}{\cot A - \operatorname{cosec} A + 1} = \frac{1 + \cos A}{\sin A}$$

$$47. \frac{1+\sin A}{\operatorname{cosec} A - \cot A} - \frac{1-\sin A}{\operatorname{cosec} A + \cot A} = 2(1 + \cot A)$$

$$48. \frac{\cos \theta \cot \theta}{1+\sin \theta} = \operatorname{cosec} \theta - 1$$

1. Prove that :

$$(i) \frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A$$

[2003]

$$(ii) \frac{\cos^3 A + \sin^3 A}{\cos A + \sin A} + \frac{\cos^3 A - \sin^3 A}{\cos A - \sin A} = 2$$

$$(iii) \frac{\tan A}{1-\cot A} + \frac{\cot A}{1-\tan A} = \sec A \operatorname{cosec} A + 1$$

$$(iv) \left(\tan A + \frac{1}{\cos A} \right)^2 + \left(\tan A - \frac{1}{\cos A} \right)^2 \\ = 2 \left(\frac{1+\sin^2 A}{1-\sin^2 A} \right)$$

$$(v) 2 \sin^2 A + \cos^4 A = 1 + \sin^4 A$$

$$(vi) \frac{\sin A - \sin B}{\cos A + \cos B} + \frac{\cos A - \cos B}{\sin A + \sin B} = 0$$

$$(vii) (\operatorname{cosec} A - \sin A)(\sec A - \cos A) \\ = \frac{1}{\tan A + \cot A}$$

$$(viii) (1 + \tan A \cdot \tan B)^2 + (\tan A - \tan B)^2 \\ = \sec^2 A \sec^2 B$$

$$(ix) \frac{1}{\cos A + \sin A - 1} + \frac{1}{\cos A + \sin A + 1} \\ = \operatorname{cosec} A + \sec A$$

2. If $x \cos A + y \sin A = m$ and

$x \sin A - y \cos A = n$, then prove that :

$$x^2 + y^2 = m^2 + n^2$$

3. If $m = a \sec A + b \tan A$ and

$n = a \tan A + b \sec A$, then prove that :

$$m^2 - n^2 = a^2 - b^2$$

4. If $x = r \sin A \cos B$, $y = r \sin A \sin B$ and $z = r \cos A$, then prove that :

$$x^2 + y^2 + z^2 = r^2$$

5. If $\sin A + \cos A = m$ and

$\sec A + \operatorname{cosec} A = n$, show that :

$$n(m^2 - 1) = 2m$$

6. If $x = r \cos A \cos B$, $y = r \cos A \sin B$ and $z = r \sin A$, show that :

$$x^2 + y^2 + z^2 = r^2$$

7. If $\frac{\cos A}{\cos B} = m$ and $\frac{\cos A}{\sin B} = n$,

show that :

$$(m^2 + n^2) \cos^2 B = n^2.$$