Integration

(1) $\int \frac{x^2-1}{x^4+1}\,dx$

$$I = \int \frac{(x^2-1)\,dx}{x^4+1}$$

$$= \int \frac{(x^2-1)/x^2}{(x^4+1)/x^2}\,dx$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2x\cdot\frac{1}{x}}$$

$$= \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-2}$$

Let $x+\frac{1}{x} = z$

$\frac{dz}{dx} = \left(1-\frac{1}{x^2}\right)$

$dz = \left(1-\frac{1}{x^2}\right)dx$

$$= \int \frac{dz}{z^2-(\sqrt{2})^2}$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C$$

$$= \frac{1}{2\sqrt{2}}\log\left|\frac{x^2+1-\sqrt{2}x}{x^2+1+\sqrt{2}x}\right| + C.$$

(2) $\int \frac{1+2x^2}{x^2(1+x^2)}\,dx$

$$\int \frac{(1+x^2)+x^2}{x^2(1+x^2)}\,dx$$

$$= \int \frac{dx}{x^2} + \int \frac{dx}{(1+x^2)}$$

Formula

$$\left[\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}x/a\right.$$

$$\int x^{-2}dx + \int \frac{dx}{x^2+1^2}$$

$$= \frac{x^{-2+1}}{-2+1} + \frac{1}{1}\tan^{-1}x/1 + C$$

$$= -\frac{1}{x} + \tan^{-1}x + C$$

(3) $\int \frac{\sin x\cos x\,dx}{\cos^4x+\sin^4x}$

$$\boxed{2(a^2+b^2) = (a+b)^2+(a-b)^2}$$

$$\int \frac{2\sin x\cos x}{2(\cos^4x+\sin^4x)}\,dx$$

$$= \int \frac{\sin 2x\,dx}{(\cos^2x+\sin^2x)^2+(\cos^2x-\sin^2x)^2}$$

$$= \int \frac{\sin 2x\,dx}{1+\cos^2 2x}$$

$$\int \frac{-dz}{2(1+z^2)}$$ Let $\cos 2x = z$

$\frac{dz}{dx} = -2\sin 2x \, dx$

$$= -\frac{1}{2}\int \frac{dz}{1+z^2}$$

$$= -\frac{1}{2}\tan^{-1}(z/1) + C$$

$$= -\frac{1}{2}\tan^{-1}\left(\frac{\cos 2x}{1}\right) + C.$$

(4) $\int \frac{2x+1}{x(x+3)}$

$$= \int \frac{(2x+3)-2}{(x^2+3x)}\,dx$$

$$= \int \frac{(2x+3)\,dx}{x^2+3x} - \int \frac{2\,dx}{x^2+3x}$$

$$\log|x^2+3x| - 2\int \frac{dx}{x^2+3x}$$

$$= \log|x^2+3x| - 2\int \frac{dx}{\left(x^2+2\cdot x\cdot\frac{3}{2}+\frac{9}{4}-\frac{9}{4}\right)}$$

$$= \log|x^2+3x| - 2\int \frac{dx}{\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2}$$

Let $x+\frac{3}{2} = z$

$dz = dx$

$$= \log|x^2+3x| - 2\int \frac{dz}{z^2-(3/2)^2}$$

$$= \log|x^2+3x| - 2\times\frac{1}{2\times 3/2}\log\left|\frac{z-3/2}{z+3/2}\right| + C$$

$$= \log|x^2+3x| - \frac{2}{3}\log\left|\frac{x}{x+3}\right| + C$$

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(5) $\int \frac{dx}{(x-1)\sqrt{x^2+4}}$

Let $x-1 = \frac{1}{z} \Rightarrow dx = -\frac{1}{z^2}dz$

$$\int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\left(\frac{1}{z}+1\right)^2+4}}$$

$$\int \frac{-\frac{1}{z^2}dz}{\frac{1}{z}\sqrt{\frac{1}{z^2}+\frac{2}{z}+1+4}}$$

$$= \int \frac{-\frac{1}{\cancel{z^2}}dz}{\frac{1}{\cancel{z}}\frac{\sqrt{5z^2+2z+1}}{\cancel{z}}}$$

$$= \int \frac{-dz}{\sqrt{5z^2+2z+1}}$$

$$= \int \frac{-dz}{\sqrt{5\left(z^2+2\cdot\frac{z}{5}+\frac{1}{25}-\frac{1}{25}\right)+\frac{1}{5}}}$$

$\frac{1}{5}-\frac{1}{25}$
$= \frac{5-1}{25}$
$= \frac{4}{25}$

$$= \int \frac{-dz}{\sqrt{5}\sqrt{\left(z+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}}$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(z+\frac{1}{5}\right)+\sqrt{\left(z+\frac{1}{5}\right)^2+\left(\frac{2}{5}\right)^2}\right|$$

$$= -\frac{1}{\sqrt{5}}\log\left|\left(\frac{1}{x-1}+\frac{1}{5}\right)+\sqrt{\left(\frac{1}{x-1}+\frac{1}{5}\right)^2+\frac{4}{25}}\right| + C$$

(6) $\int \frac{dx}{(x-1)\sqrt{x+3}}$

$x+3 = z^2$

$2z\,dz = dx$

$\int \frac{2\cancel{z}\,dz}{(z^2-4)\cancel{z}}$

$= 2\int \frac{dz}{z^2-4}$

$= 2\int \frac{dz}{z^2-2^2}$

$= \cancel{2} \times \frac{1}{2\times\cancel{2}} \log\left|\frac{z-2}{z+2}\right| + C$

$= \frac{1}{2} \log\left|\frac{\sqrt{x+3}-2}{\sqrt{x+3}+2}\right| + C$

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