Lagranges Theo

1 $f(x) = x(x+4)^2 \quad 0 \le x \le 4$

$f(x) = x(x^2+8x+16)$

$f(x) = x^3+8x^2+16x$

$f(x)$ is polynomial

$\therefore f(x)$ is continuous in $0 \le x \le 4$

$\frac{d}{dx} f(x) = 3x^2+16x+16$

$f'(x) = 3x^2+16x+16$

$\therefore f$ is derivable in $[0,4]$

$f(4) = 4(4+4)^2 = 256$

$f(0) = 0$

To verify lagranges mean value Theo,

$$\frac{f(4)-f(0)}{4-0} = f'(c)$$

$$\frac{\cancel{256}^{64}}{\cancel{4}} = 3c^2+16c+16$$

$$3c^2+16c+16-64=0$$

$$3c^2+16c-48=0$$

$$c = \frac{-16 \pm \sqrt{256-4\times3\times(-48)}}{2\times3}$$

$$= \frac{-16 \pm \sqrt{256+576}}{6}$$

$$= \frac{-16 \pm \sqrt{832}}{6}$$

$$= \frac{-16 \pm 8\sqrt{13}}{6} = \frac{4\sqrt{13}-8}{16} \in (0,4)$$

(+ve is taken) Hence verified

(2)

Find a point on the parabola $y=(x-2)^2$ where the tangent is ∥ to the chord joining $(2,0)$ and $(3,1)$

Ans

$(x_1, y_1) = (2,0)$

$(x_2, y_2) = (3,1)$

$f(x) = y = (x-2)^2$, $[2,3]$

Applying Lagranges Theo,

$f(x) = (x-2)^2$

$f(x) = x^2-4x+4$

$\frac{dy}{dx} = 2x-4$

$f'(c) = 2c-4$

$f(2) = 0$

$f(3) = (3-2)^2 = 1^2$

$$\frac{f(3)-f(2)}{3-2} = f'(c)$$

$$\frac{1-0}{1} = 2c-4$$

$2c = 4+1$

$c = \frac{5}{2}$

$c = 2\frac{1}{2} \in (2,3)$

~~Note~~ Now value of y when $x = 5/2$ is

$$y = \left(\frac{5}{2}-2\right)^2 = \frac{1}{4}$$

∴ point on which the tangent is parallel to the chord joining two given point is $\left(\frac{5}{2}, \frac{1}{4}\right)$

3) Let the function, $f: [-7,0] \to R$ be continuous on $[-7,0]$ and differentiable on $(-7,0)$. If $f(-7) = -3$ and $f'(x) \le 2$ for all $x \in (-7,0)$, then for all such functions f, $f(-1) + f(0)$ lies in the in interval

$$f(-7) = -3$$

$$f(0) = K \text{ (say)}$$

$$\cancel{\frac{f(-7) - f(0)}{6}} \quad \frac{f(0) - f(-7)}{0-(-7)} = f'(c) \le 2$$

$$\frac{f(0)+3}{7} \le 2$$

$$f(0) \le 14 - 3$$

$$f(0) \le 11 \quad \text{--- (1)}$$

$$\cancel{\frac{f(0) - f(-7)}{0-(}} \quad \frac{f(-1) - f(-7)}{-1-(-7)} = f'(c) \le 2$$

$$\frac{f(-1)+3}{6} \le 2$$

$$f(-1) + 3 \le 12$$

$$f(-1) \le 9$$

$$f(-1) + f(0) \le 11 + 9$$

$$\le 20$$

4. Use Lagranges mean value theo to find a point P on the curve $y=\sqrt{x^2-4}$ defined in the interval $(2,4)$ where the tangent is parallel to the chord Joining the end end-points on the curve.

Ans (i) $f(x)$ is continuous in $[2,4]$

$y=f(x)=\sqrt{x^2-4}$ (ii) $\frac{d}{dx}f(x)=\frac{1}{\cancel{2}\sqrt{x^2-4}}\cdot\cancel{2}x$

$\therefore f$ is derivable.

$f(2)=0$

$f(4)=\sqrt{16-4}=\sqrt{12}=2\sqrt{3}$.

To verify Lagranges theo:

Let $\frac{f(4)-f(2)}{4-2}=f'(c)$

$\frac{2\sqrt{3}-0}{2}=\frac{c}{\sqrt{c^2-4}}$

$\sqrt{3}=\frac{c}{\sqrt{c^2-4}}$

$3c^2-12=c^2$

$2c^2=12$

$c^2=6$

$c=\sqrt{6}$

$P=(\sqrt{6},\sqrt{2})$

(5) Find a point on the curve $y=x^3$ where the tangent is parallel to the chord joining the points $\underset{x_1\ y_1}{(0,0)}$ and $\underset{x_2\ y_2}{(3,27)}$

$y=f(x)=x^3$

$\therefore f(x)=x^3$ is continuous in $[x_1,x_2]=[0,3]$

$\frac{d}{dx}f(x)=3x^2$

$\therefore f(x)$ is derivable in $(0,3)$

$f(0)=0$

$f(3)=27$

To verify Lagrange's mean value theo:

$\frac{f(3)-f(0)}{3-0}=f'(c)$

$\frac{27-0}{3}=3c^2$

$9=3c^2$

$c^2=3$

$c=\sqrt{3}$

$f(c)=(\sqrt{3})^3=3\sqrt{3}$

point where tangent is $\parallel$ to chord is $(\sqrt{3},3\sqrt{3})$

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Inverse

(1) prove that $\tan^{-1} 4/3 + \tan^{-1} \frac{12}{5} = \pi - \tan^{-1} \frac{56}{33}$

Formula:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy} \quad \text{if } xy < 1$$

$x>0, y>0$

$$= \pi - \tan^{-1} \frac{x+y}{1-xy} \quad \text{if } xy > 1$$

Here $x = 4/3$

$$y = \frac{12}{5}$$

$$xy = \frac{4}{\cancel{3}} \times \frac{\cancel{12}^{4}}{5} = \frac{16}{5} > 1$$

$\therefore$ $\tan^{-1} 4/3 + \tan^{-1} \frac{12}{5}$

$$= \pi + \tan^{-1} \frac{\frac{4}{3} + \frac{12}{5}}{1 - \frac{4}{3} \times \frac{12}{5}}$$

$$= \pi + \tan^{-1} \left(\frac{\frac{20+36}{15}}{\frac{15-48}{15}} \right)$$

$$= \pi + \tan^{-1} \left(\frac{56}{-33} \right)$$

$$= \pi - \tan^{-1} \frac{56}{33}$$

proved

(2) Show that $\cos\left[\cos^{-1}x + \sin^{-1}(x-2)\right] = 0$

$\cos^{-1}x$ is defined if

$-1 \le x \le 1$ —— (i)

$\sin^{-1}(x-2)$ is defined if

$-1 \le x-2 \le 1$

$1 \le x \le 3$ —— (ii)

x satisfies both (1) & (ii) if $x=1$

$\cos\left[\cos^{-1}1 + \sin^{-1}(1-2)\right]$

$= \cos\left[\cos^{-1}1 + \sin^{-1}(-1)\right]$

$= \cos\left[0 - \pi/2\right]$

$= 0$ proved.

(8) $\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1}\left(\frac{x}{2-x^2}\right) = \cos^{-1}K + \pi$

Then find K

$\sin^{-1}(x-1)$ is defined if

$$-1 \le x-1 \le 1$$
$$0 \le x \le 2 \quad \text{—— (i)}$$

$\cos^{-1}(x-3)$ is defined if

$$-1 \le x-3 \le 1$$
$$2 \le x \le 4 \quad \text{—— (ii)}$$

from (i) & (ii)

$$x = 2$$

Now

$$\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1}\left(\frac{x}{2-x^2}\right) = \cos^{-1}K + \pi$$

$$\Rightarrow \sin^{-1}(1) + \cos^{-1}(-1) + \tan^{-1}\left(\frac{2}{2-4}\right) = \cos^{-1}K + \pi$$

$$\Rightarrow \frac{\pi}{2} + \cancel{\pi} + \left(-\frac{\pi}{4}\right) = \cos^{-1}K + \cancel{\pi}$$

$$\frac{\pi}{4} = \cos^{-1}K$$

$$K = \cos\pi/4 = \frac{1}{\sqrt{2}}$$

4 Prove that $2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}$

Let $\cancel{2}\tan^{-1}x = \theta$

$\tan\theta = x$

Again ~~we know~~ $\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

$\sin 2\theta = \frac{2x}{1+x^2}$

$2\theta = \sin^{-1}\frac{2x}{1+x^2}$

$\theta = \frac{1}{2}\sin^{-1}\frac{2x}{1+x^2}$

5 Find the simplest value of $2\cot^{-1}\left(\frac{1-x}{1+x}\right) + \csc^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

Ans Put $x = \tan\theta$

$2\cot^{-1}\left(\frac{1-x}{1+x}\right) + \csc^{-1}\left(\frac{1+x^2}{1-x^2}\right)$

$= 2\cot^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) + \csc^{-1}\left(\frac{1+\tan^2\theta}{1-\tan^2\theta}\right)$

$= \cancel{2\cot^{-1}\tan}$

$= 2\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right) + \sin^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= 2\tan^{-1}\tan\left(\frac{\pi}{4}+\theta\right) + \sin^{-1}\cos 2\theta$

$= 2\left(\pi/4+\theta\right) + \sin^{-1}\sin\left(\pi/2-2\theta\right)$

$= \pi/2 + 2\theta + \pi/2 - 2\theta$

$= \pi$.

(6) $\sec\theta - \csc\theta = 4/3$

Show that $\theta = \frac{1}{2}\sin^{-1}3/4$

$$\sec\theta - \csc\theta = \frac{4}{3}$$

$$\Rightarrow \frac{1}{\cos\theta} - \frac{1}{\sin\theta} = 4/3$$

$$\Rightarrow \frac{\sin\theta - \cos\theta}{\sin\theta\cos\theta} = 4/3$$

$$\Rightarrow 3(\sin\theta - \cos\theta) = 4\sin\theta\cos\theta$$

$$\Rightarrow 3(\sin\theta - \cos\theta) = 2\sin 2\theta.$$

$$\Rightarrow 9(\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta) = 4\sin^2 2\theta$$

$$\Rightarrow 9(1 - \sin 2\theta) = 4\sin^2 2\theta$$

$$4\sin^2 2\theta + 9\sin 2\theta - 9 = 0$$

$$\sin 2\theta = \frac{-9 \pm \sqrt{81 + 4\times 4\times 9}}{2\times 4}$$

$$= \frac{-9 \pm \sqrt{225}}{8}$$

$$= \frac{-9 \pm 15}{8}$$

$$\sin 2\theta = \frac{6}{8} = \frac{3}{4} \qquad \Big| \qquad \sin 2\theta = -3$$

$$2\theta = \sin^{-1}\left(\frac{3}{4}\right)$$

$$\theta = \frac{1}{2}\sin^{-1}(3/4)$$ proved.

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7. Solve

$\sin^{-1}x + \sin^{-1} 2x = \pi/3$

$$\sin^{-1} 2x = (\pi/3 - \sin^{-1}x)$$

$$2x = \sin(\pi/3 - \sin^{-1}x)$$

$$2x = \sin\pi/3 \cos\sin^{-1}x - \sin\sin^{-1}x \cos\pi/3$$

$$\Rightarrow 2x = \frac{\sqrt{3}}{2} \cos\cos^{-1}\sqrt{1-x^2} - \frac{x}{2}$$

$$\Rightarrow 2x + \frac{x}{2} = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$\Rightarrow \frac{5x}{\cancel{2}} = \frac{\sqrt{3}\sqrt{(1-x^2)}}{\cancel{2}}$$

$$\Rightarrow \frac{25x^2}{\cancel{4}} = \frac{3(1-x^2)}{\cancel{4}}$$

$$\Rightarrow 28x^2 = 3$$

$$x^2 = \frac{3}{28}$$

$$x = \frac{\sqrt{3}}{2\sqrt{7}}$$

$$x = \frac{\sqrt{3}\times\sqrt{7}}{14} = \frac{\sqrt{21}}{14}$$

(8) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$

$xy + yz + zx = 1$

Ans $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$

$$\tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx} = \pi/2$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx} = \tan(\pi/2)$$

$$\Rightarrow 1-xy-yz-zx = 0$$

$$1 = xy + yz + zx \quad \text{Pr}$$

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